

# Frames and numerical approximation – supplementary material

*Ben Adcock and Daan Huybrechs*

*Report TW 675, December 2016*



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## **Abstract**

This document contains supplementary materials for the paper *Frames and numerical approximation* by B. Adcock & D. Huybrechs (technical report TW-674).

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# Frames and numerical approximation – supplementary materials

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## Abstract

This document contains supplementary materials for the paper *Frames and numerical approximation* by B. Adcock & D. Huybrechs [3].

## SM1 Example 1. Fourier frames for complex geometries

Consider the frame (3.1) over a domain  $\Omega \subseteq (-1, 1)^d$ .

### SM1.1 The kernel of $\mathcal{G}$

We first characterize the kernel of the Gram operator:

**Proposition SM1.1.** *Let  $\mathcal{G}$  be the Gram operator (2.7) of the frame (3.1). Then*

$$\text{Ker}(\mathcal{G}) = \left\{ \{\hat{f}_n\}_{n \in \mathbb{Z}^d} : f \in L^2(-1, 1)^d, f(x) = 0 \text{ a.e. } x \in \Omega \right\},$$

where  $\hat{f}_n = \int_{(-1, 1)^d} f(t) \overline{\phi_n(t)} dt$  are the Fourier coefficients of  $f \in L^2(-1, 1)^d$ .

*Proof.* Let  $f \in L^2(-1, 1)^d$  with  $f|_\Omega = 0$ . Let  $x = \{\hat{f}_n\}_{n \in \mathbb{Z}^d}$ . Then

$$\mathcal{G}x = \{\langle f, \phi_n \rangle\}_{n \in \mathbb{Z}^d} = 0.$$

Conversely, if  $x \in \text{Ker}(\mathcal{G})$  then  $x = \{\hat{f}_n\}_{n \in \mathbb{Z}^d}$  for some  $f \in L^2(-1, 1)^d$ . Notice that

$$0 = x^* \mathcal{G}x = \left\| \sum_{n \in \mathbb{Z}^d} x_n \phi_n \right\|^2 = \int_\Omega |f(x)|^2 dx.$$

Hence  $f(x) = 0$  a.e. for  $x \in \Omega$ . □

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## SM1.2 Proofs of Propositions 4.5 and 5.8

We first require some notation. Let  $H^k(\Omega)$  be the  $k^{\text{th}}$  standard Sobolev space of functions on  $\Omega$ , and  $H_0^k(\Omega)$  be the closure of  $C_c^\infty(\Omega)$  with respect to the corresponding norm  $\|\cdot\|_{H^k(\Omega)}$ . Throughout this section, we also let  $c_{k,d}$  denote an arbitrary constant that is independent of  $N$  and of the particular function  $f$  being approximated.

The following lemma confirms the existence of small norm extensions:

**Lemma SM1.2.** *Let  $\Omega \subseteq (-1, 1)^d$  be a Lipschitz domain. Then for any  $k \in \mathbb{N}$  there exists a linear extension operator  $\mathcal{E} : H^k(\Omega) \rightarrow H_0^k(-1, 1)^d$  satisfying  $\mathcal{E}f(x) = f(x)$  a.e. for  $x \in \Omega$  and*

$$\|\mathcal{E}f\|_{H^k(-1, 1)^d} \leq c_{k,d} \|f\|_{H^k(\Omega)}. \quad (\text{SM1.1})$$

*Proof.* Recall that there exists an extension operator  $\mathcal{E}' : H^k(\Omega) \rightarrow H^k(-1, 1)^d$  satisfying  $\mathcal{E}'f(x) = f(x)$  a.e. for  $x \in \Omega$  and  $\|\mathcal{E}'f\|_{H^k(-1, 1)^d} \leq c_{k,d} \|\mathcal{E}'f\|_{H^k(\Omega)}$  some positive constant  $c_{k,d}$  [1]. Moreover, there also exists a smooth bump function  $g \in C_0^\infty(\mathbb{R}^d)$  satisfying  $g(x) = 1$ ,  $x \in \Omega$  and  $g(x) = 0$ ,  $x \notin (-1, 1)^d$ . We claim that the operator  $\mathcal{E}f(x) = g(x)\mathcal{E}'f(x)$  is a suitable extension operator. Certainly  $\mathcal{E}f(x) = f(x)$ , a.e. for  $x \in \Omega$  and  $\mathcal{E}f(x) = 0$  outside  $(-1, 1)^d$ . Also  $\mathcal{E}f \in H^k(\Omega)$  by construction. Finally, a simple calculation confirms that (SM1.1) holds for some suitable constant  $c_{k,d}$ .  $\square$

In order to prove results about this frame, we need to recall some basic Fourier analysis. In particular, the following result is standard (see [5, Chpt. 5]):

**Lemma SM1.3.** *Let  $g \in H_0^{kd}(-1, 1)^d$  and consider its Fourier coefficients  $\hat{g}_n = \int_{(-1, 1)^d} g(t) \overline{\phi_n(t)} dt$ , where  $\phi_n(t)$  is as in (3.1). If  $I_N$  is as in (3.2) then*

$$\left\| g - \sum_{n \notin I_N} \hat{g}_n \phi_n \right\|_{L^2(-1, 1)^d} \leq c_{k,d} N^{-k} \|g\|_{H^{kd}(-1, 1)^d}.$$

Note that it is not necessary for  $g$  to vanish on the boundary of  $(-1, 1)^d$  for this lemma to hold. Instead, it needs to belong to the periodic Sobolev space  $H_p^{kd}(-1, 1)^d$  over the unit torus  $(-1, 1)^d$  [5]. However, all construction we consider will yield functions in  $H_0^{kd}(-1, 1)^d$ . Hence we state the result as above. With this to hand, we can now prove Proposition 5.8:

*Proof of Proposition 5.8.* By Lemma SM1.2 there is an extension  $g \in H_0^{kd}(-1, 1)^d$  of  $f$  with  $\|g\|_{H^{kd}(-1, 1)^d} \leq c_{k,d} \|f\|_{H^{kd}(\Omega)}$ . Let  $\hat{g}_n$  be the Fourier coefficients of  $g$  on  $(-1, 1)^d$  and set  $x_n = \hat{g}_n$  for  $n \in I = \mathbb{Z}^d$ . Then, by Parseval's formula for the Fourier basis on  $(-1, 1)^d$ , we have  $\|x\| = \|g\|_{L^2(-1, 1)^d} \leq c_{k,d} \|f\|_{H^{kd}(\Omega)}$ . Furthermore,  $\|f - \mathcal{T}_N x\| \leq \|g - \mathcal{T}_N x\|_{L^2(-1, 1)^d}$ . But  $\mathcal{T}_N x$  is just the partial Fourier series of the function  $g$  on  $(-1, 1)^d$ . Hence, by Lemma SM1.3

$$\|f - \mathcal{T}_N x\| \leq c_{k,d} N^{-k} \|g\|_{H^{kd}(-1, 1)^d} \leq c_{k,d} N^{-k} \|f\|_{H^{kd}(\Omega)},$$

which gives the result.  $\square$

We are now able to prove Proposition 4.5:

*Proof of Proposition 4.5.* Define the coefficients  $x_n$ ,  $n \in \mathbb{Z}^d$ , by  $x_0 = 1$  and  $x_n = 0$  otherwise. Then (4.2) gives that  $B_N \geq \|\phi_0\|^2 = 2^{-d} \text{Vol}(\Omega) \gtrsim 1$ . It therefore suffices to consider the lower frame

bound  $A_N$ . Let  $g \in H_0^{kd}(-1, 1)^d$  be such that  $\|g\|_{L^2(-1, 1)^d} = 1$  and  $g(x) = 0$  a.e. for  $x \in \Omega$ . Suppose that  $x \in \ell^2(I)$  is the vector of Fourier coefficients of  $g$ . Then

$$A_N \leq \frac{\|\mathcal{T}_N x\|^2}{\|x\|^2} = \frac{\|g - \mathcal{T}_N x\|^2}{\|g\|_{L^2(-1, 1)^d}^2} \leq c_{k,d}^2 N^{-2k} \|g\|_{H^{kd}(-1, 1)^d}^2,$$

by Lemma SM1.3. □

## SM2 Example 2. Augmented Fourier basis

We now consider the augmented Fourier frame (3.3).

### SM2.1 The kernel of $\mathcal{G}$

We first characterize the kernel of the Gram operator:

**Proposition SM2.1.** *Let  $\mathcal{G}$  be the Gram operator (2.7) of the frame (3.3). Then  $\mathcal{G}$  consists of vectors of the form*

$$\begin{bmatrix} \{\langle p, \psi_k \rangle\}_{k=1}^K \\ \{\langle -p, \varphi_n \rangle\}_{n \in \mathbb{Z}} \end{bmatrix}, \quad p \in \mathbb{P}_K^0. \quad (\text{SM2.1})$$

In particular,  $\text{Ker}(\mathcal{G})$  has dimension  $K$ .

*Proof.* Let  $x$  be of the form (SM2.1). Then

$$\mathcal{T}x = \sum_{k=1}^K \langle p, \psi_k \rangle \psi_k + \sum_{n \in \mathbb{Z}} \langle -p, \varphi_n \rangle \varphi_n = p - p = 0.$$

Hence  $\mathcal{G}x = \mathcal{T}^* \mathcal{T}x = 0$ . Conversely, let  $x \in \text{Ker}(\mathcal{G})$ . We may write  $x = \begin{bmatrix} \{\langle p, \psi_k \rangle\}_{k=1}^K \\ \{\langle f, \varphi_n \rangle\}_{n \in \mathbb{Z}} \end{bmatrix}$ , where  $p \in \mathbb{P}_K^0$  and  $f \in L^2(-1, 1)$ . Then

$$x^* \mathcal{G}x = \|\mathcal{T}x\|^2 = \left\| \sum_{k=1}^K \langle p, \psi_k \rangle \psi_k + \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n \right\|^2 = \|p + f\|^2.$$

Hence  $f = -p$  and therefore  $x$  is of the form (SM2.1). □

### SM2.2 The canonical dual frame

Although this frame is not tight, its canonical dual frame has an explicit expression. Let  $\mathcal{Q}_K$  and  $\mathcal{F}_{N-K}$  be the orthogonal projections onto  $\text{span}\{\psi_1, \dots, \psi_K\}$  and  $\text{span}\{\varphi_n : n = -\frac{N-K}{2}, \dots, \frac{N-K}{2} - 1\}$  respectively. Then we have the following:

**Proposition SM2.2.** *The frame operator of the frame (3.3) satisfies*

$$\mathcal{S} = \mathcal{I} + \mathcal{Q}_K, \quad \mathcal{S}^{-1} = \mathcal{I} - \frac{1}{2} \mathcal{Q}_K. \quad (\text{SM2.2})$$

Furthermore, the truncated canonical dual frame expansion  $f_N = \sum_{n \in I_N} \langle \mathcal{S}^{-1} f, \phi_n \rangle \phi_n$  satisfies

$$f - f_N = (\mathcal{I} - \mathcal{F}_{N-K}) \mathcal{S}^{-1} f. \quad (\text{SM2.3})$$

*Proof.* Observe that

$$\mathcal{S} = \sum_{n \in \mathbb{Z}} \langle \cdot, \varphi_n \rangle \varphi_n + \sum_{k=1}^K \langle \cdot, \psi_k \rangle \psi_k = \mathcal{I} + \mathcal{Q}_K,$$

and therefore

$$\left( \mathcal{I} - \frac{1}{2} \mathcal{Q}_K \right) (\mathcal{I} + \mathcal{Q}_K) = \mathcal{I} - \frac{1}{2} \mathcal{Q}_K + \mathcal{Q}_K - \frac{1}{2} \mathcal{Q}_K^2 = \mathcal{I},$$

since  $\mathcal{Q}_K$  is a projection. This gives (SM2.2).

Consider (SM2.3). The truncated canonical dual frame expansion is

$$f_N = \mathcal{F}_{N-K} \mathcal{S}^{-1} f + \mathcal{Q}_K \mathcal{S}^{-1} f = \mathcal{F}_{N-K} f - \frac{1}{2} \mathcal{F}_{N-K} \mathcal{Q}_K f + \frac{1}{2} \mathcal{Q}_K f,$$

as required.  $\square$

The expression (SM2.3) shows that the tail of the canonical dual frame expansion is equal to the tail of the Fourier expansion of the function  $\mathcal{S}^{-1} f = f - \frac{1}{2} \mathcal{Q}_K f$ . In general, this cannot converge quickly as  $N \rightarrow \infty$  when  $f$  is nonperiodic. Indeed, if  $f$  is nonperiodic then the function  $\mathcal{S}^{-1} f$  is also nonperiodic. Nor can it be well approximated by a periodic function. In particular, for large  $K$  one has  $\mathcal{S}^{-1} f \approx \frac{1}{2} f$ , which is just a scaled version of  $f$ .

### SM2.3 Proofs of Propositions 4.6 and 5.9

To prove these results, we shall first exploit the fact that the Fourier coefficients of a smooth but nonperiodic function  $f$  can be written as the sum of Fourier coefficients of a polynomial, plus a remainder term which is asymptotically small. This is rather standard approach in the analysis of Fourier expansions [2, 4, 6, 7].

We first require the following lemma:

**Lemma SM2.3.** *Let  $K \in \mathbb{N}$  and suppose that  $c_0, \dots, c_{K-1} \in \mathbb{C}$ . Then there exists a unique polynomial  $p \in \mathbb{P}_K^0$  satisfying  $p^{(r)}(1) - p^{(r)}(-1) = c_r$  for  $r = 0, \dots, K-1$ .*

*Proof.* For  $r = 0, 1, \dots$ , let  $p_r(t) = \frac{2^r}{(r+1)!} B_{r+1}((t+1)/2)$ , where  $B_{r+1} \in \mathbb{P}_{r+1}$  is the  $r$ th Bernoulli polynomial. By properties of the Bernoulli polynomials, we have  $\int_{-1}^1 p_r(t) dt = 0$  and  $p_s^{(r)}(1) - p_s^{(r)}(-1) = \delta_{r,s}$ ,  $r, s = 0, \dots, K-1$ . Hence, such a  $p \in \mathbb{P}_K$  exists and can be written as  $p(t) = \sum_{r=0}^{K-1} c_r p_{r+1}(t)$ . For uniqueness, we note that the system  $\{1, p_1, p_2, \dots, p_{K-1}\}$  forms a basis for the space  $\mathbb{P}_{K-1}$ .  $\square$

Consider the Fourier coefficient  $\langle f, \varphi_n \rangle$  of a function  $f$ , where  $\varphi_n(t) = \frac{1}{\sqrt{2}} e^{in\pi t}$ . If  $f \in H^k(-1, 1)$  and  $n \neq 0$  then integrating by parts  $k$  times gives

$$\begin{aligned} \langle f, \varphi_n \rangle &= \frac{1}{\sqrt{2}} \int_{-1}^1 f(t) e^{-in\pi t} dt \\ &= \frac{(-1)^{n+1}}{\sqrt{2}} \sum_{r=0}^{k-1} \frac{f^{(r)}(1) - f^{(r)}(-1)}{(in\pi)^{r+1}} + \frac{1}{(in\pi)^k} \langle f^{(k)}, \varphi_n \rangle. \end{aligned}$$

In particular, for the polynomials  $p_r(t)$  introduced in the above proof, we have

$$\langle p_r, \varphi_n \rangle = \frac{(-1)^{n+1}}{\sqrt{2}(in\pi)^{r+1}}, \quad \langle p_r, \varphi_0 \rangle = 0 \quad (\text{SM2.4})$$

Hence, if  $f \in H^K(-1, 1)$  and  $p \in \mathbb{P}_K$  is the polynomial with

$$p^{(r)}(1) - p^{(r)}(-1) = f^{(r)}(1) - f^{(r)}(-1), \quad r = 0, \dots, K-1,$$

then we have

$$\langle f, \varphi_n \rangle = \langle p, \varphi_n \rangle + \frac{1}{(in\pi)^k} \langle f^{(k)}, \varphi_n \rangle. \quad (\text{SM2.5})$$

We may now prove Proposition 5.9:

*Proof of Proposition 5.9.* Since  $f$  has  $k-1$  continuous derivatives, Lemma SM2.3 ensures the existence of a polynomial  $p \in \mathbb{P}_k^0$  with  $p^{(r)}(1) - p^{(r)}(-1) = f^{(r)}(1) - f^{(r)}(-1)$  for  $r = 0, \dots, k-1$ . Write  $p = \sum_{r=1}^k a_r \psi_r$  and define the coefficients  $x \in \ell^2(I)$  so that

$$\mathcal{T}x = \sum_{n \in \mathbb{Z}} \langle f - p, \varphi_n \rangle \varphi_n + \sum_{k=1}^K a_k \psi_k,$$

Note that

$$\|x\|^2 = \sum_{n \in \mathbb{Z}} |\langle f - p, \varphi_n \rangle|^2 + \|a\|^2 = \|f - p\|^2 + \|a\|^2.$$

It is straightforward to see that there exists a constant  $c_k$  such that

$$\|a\|, \|p\| \leq c_k \max_{r=0, \dots, k-1} |f^{(r)}(1) - f^{(r)}(-1)| \leq \sqrt{2} c_k \|f\|_{H^k},$$

where in the last step we use the fact that  $f^{(r)}(1) - f^{(r)}(-1) = \int_{-1}^1 f^{(r+1)}(t) dt$ . Moreover, by Parseval's theorem and (SM2.5),

$$\begin{aligned} \|f - \mathcal{T}_N x\|^2 &= \left\| f - p - \sum_{n=-\frac{N-K}{2}}^{\frac{N-K}{2}-1} \langle f - p, \varphi_n \rangle \phi_n \right\|^2 \\ &\leq \sum_{|n| \geq \frac{N-K}{2}} |\langle f - p, \varphi_n \rangle|^2 \\ &\leq \frac{1}{((N-K)\pi/2)^{2k}} \|f^{(k)}\|^2. \end{aligned}$$

This now gives the result.  $\square$

Finally, give the proof of Proposition 4.6:

*Proof of Proposition 4.6.* Since the frame  $\Phi$  contains an orthonormal basis, we have  $B_N \geq 1$ . Conversely, let  $p = p_{K-1}$  be as in the proof of Proposition SM2.3. Write  $p = \sum_{k=1}^K z_k \psi_k$  for some coefficients  $z_k$ , and let  $y_n = -\langle p, \varphi_n \rangle$ ,  $n \in \mathbb{Z}$ . Then

$$A_N = \min_{x \in \mathbb{C}^N, \|x\|=1} x^* G_N x \leq \frac{\left\| p - \sum_{n=-\frac{N-K}{2}}^{\frac{N-K}{2}-1} \langle p, \varphi_n \rangle \phi_n \right\|^2}{\sum_{n=-\frac{N-K}{2}}^{\frac{N-K}{2}-1} |\langle p, \varphi_n \rangle|^2} \leq \frac{E_N}{\|p\|^2 - E_N},$$

where  $E_N = \sum_{|n| \geq \frac{N-K}{2}} |\langle p, \varphi_n \rangle|^2$ . Using (SM2.4), we deduce that  $E_N \lesssim N^{1-2K}$  for  $N > K$ .  $\square$

### SM3 Example 3. Polynomials plus modified polynomials

We now consider the frame (3.6).

#### SM3.1 The kernel of $\mathcal{G}$

**Proposition SM3.1.** *Let  $\mathcal{G}$  be the Gram operator (2.7) of the frame (3.6). Then  $\text{Ker}(\mathcal{G})$  consists of vectors of the form*

$$\begin{bmatrix} \{\langle f, \varphi_n \rangle\}_{n \in \mathbb{N}} \\ -c \end{bmatrix}, \quad (\text{SM3.1})$$

where  $f \in L^2(-1, 1)$  is arbitrary and  $c \in \ell^2(\mathbb{N})$  is any vector satisfying  $\sum_{n \in \mathbb{N}} c_n \psi_n = f$ , where  $\psi_n = w\varphi_n$ .

*Proof.* Let  $x$  be of the form (SM3.1). Then

$$\mathcal{T}x = \sum_{n \in \mathbb{N}} \langle f, \varphi_n \rangle \varphi_n - \sum_{n \in \mathbb{N}} c_n \psi_n = f - f = 0.$$

Hence  $\mathcal{G}x = \mathcal{T}^* \mathcal{T}x = 0$ . Conversely, let  $x \in \text{Ker}(\mathcal{G})$  and write  $x = \begin{bmatrix} \{\langle f, \varphi_n \rangle\}_{n \in \mathbb{N}} \\ -c \end{bmatrix}$  for some  $f \in L^2(-1, 1)$  and  $c \in \ell^2(\mathbb{N})$ . Then

$$x^* \mathcal{G}x = \|\mathcal{T}x\|^2 = \left\| f - \sum_{n \in \mathbb{N}} c_n \psi_n \right\|^2.$$

Hence  $c$  satisfies  $\sum_{n \in \mathbb{N}} c_n \psi_n = f$ . □

#### SM3.2 The canonical dual frame

As with the previous example, we can give an explicit expression for the canonical dual frame in this case. To this end, let  $\mathcal{Q}_N$  be the orthogonal projection onto  $\text{span}\{\varphi_1, \dots, \varphi_{N/2}\} = \mathbb{P}_{N/2-1}$ . Then

**Proposition SM3.2.** *The frame operator of the frame (3.6) satisfies*

$$\mathcal{S}f = v^{-1}f, \quad \mathcal{S}^{-1} = vf, \quad (\text{SM3.2})$$

where  $v = (1+w^2)^{-1}$ . Furthermore, the truncated canonical dual frame expansion  $f_N = \sum_{n \in I_N} \langle \mathcal{S}^{-1}f, \phi_n \rangle \phi_n$  satisfies

$$f - f_N = (\mathcal{I} - \mathcal{Q}_N)(vf) + w(\mathcal{I} - \mathcal{Q}_N)(wvf). \quad (\text{SM3.3})$$

*Proof.* Notice that if  $f$  is in  $L^2(-1, 1)$  then so is the function  $wf$ , since  $w \in L^\infty(-1, 1)$ . Hence

$$\mathcal{S}f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n + w \sum_{n=1}^{\infty} \langle f, w\varphi_n \rangle \varphi_n = (1 + w^2) f,$$

which immediately gives (SM3.2). Therefore

$$f_N = \sum_{n=1}^N \langle \mathcal{S}^{-1}f, \varphi_n \rangle \varphi_n + w \sum_{n=1}^N \langle \mathcal{S}^{-1}f, w\varphi_n \rangle \varphi_n = \mathcal{Q}_N(vf) + w\mathcal{Q}_N(wvf).$$



This gives

$$f - f_N = vf + w^2vf - \mathcal{Q}_N(vf) - w\mathcal{Q}_N(wvf) = (\mathcal{I} - \mathcal{Q}_N)(vf) + w(\mathcal{I} - \mathcal{Q}_N)(wvf),$$

as required.  $\square$

Note that the functions  $vf$  and  $wvf$  are both nondifferentiable at  $t = -1$  when  $w(t)$  is given by (3.5). Hence the convergence rate of the projections  $\mathcal{Q}_N(vf)$  and  $\mathcal{Q}_N(wvf)$  onto the space  $\mathbb{P}_{N/2-1}$  is algebraic in  $N$  with small index.

### SM3.3 Proofs of Propositions 4.7 and 5.10

*Proof of Proposition 4.7.* Since the frame  $\Phi$  contains an orthonormal basis, we have  $B_N \geq 1$ . Moreover,

$$A_N = \min_{x \in \mathbb{C}^N, \|x\|=1} x^* G_N x = \min_{\substack{p, q \in \mathbb{P}_{N/2-1} \\ \|p\| + \|q\| \neq 0}} \frac{\|p + wq\|^2}{\|p\|^2 + \|q\|^2}.$$

Set  $q(t) = (1+t)^{N/2-1} \in \mathbb{P}_{N/2-1}$  and write  $r(t) = w(t)q(t)$ . Let  $p(t) = \sum_{n=1}^{N/2} \langle r, \varphi_n \rangle \varphi_n$  be the orthogonal projection of  $r$  in the Legendre polynomial basis. Since  $\|q\|^2 = \frac{2^{N-1}}{N-1}$  and  $\|r\|^2 = \frac{2^{N+2\alpha-1}}{N+2\alpha-1}$  we have

$$A_N \leq \frac{\|r - p\|^2}{\|r - p\|^2 + \frac{2^{N-1}}{N-1} + \frac{2^{N+2\alpha-1}}{N+2\alpha-1}}. \quad (\text{SM3.4})$$

We now examine  $\|r - p\|^2$ . Using [5, (5.4.13)] we deduce that

$$\begin{aligned} \|r - p\|^2 &\leq \frac{\Gamma(N/2 - k)}{\Gamma(N/2 + k)} \int_{-1}^1 |r^{(k)}(t)|^2 (1-t^2)^k dt \\ &= \frac{\Gamma(N/2 - k)}{\Gamma(N/2 + k)} \left( \frac{\Gamma(N/2 + \alpha)}{\Gamma(N/2 + \alpha - k)} \right)^2 \int_{-1}^1 (1+t)^{N+2\alpha-2k-2} (1-t^2)^k dt, \end{aligned}$$

for  $0 \leq k \leq N/2$ . Setting  $k = N/2$  and noting that  $\int_{-1}^1 (1+t)^{2\alpha-2} (1-t^2)^{N/2} dt \leq 1$  for  $N \geq 4$  now gives

$$\|r - p\|^2 \leq \frac{1}{\Gamma(N)} \left( \frac{\Gamma(N/2 + \alpha)}{\Gamma(\alpha)} \right)^2.$$

Stirling's formula now gives

$$\frac{1}{\Gamma(N)} \left( \frac{\Gamma(N/2 + \alpha)}{\Gamma(\alpha)} \right)^2 \sim \frac{2\sqrt{\pi}}{\Gamma(\alpha)^2 (2e)^\alpha} N^{2\alpha-1/2} 2^{-N}, \quad N \rightarrow \infty.$$

The result now follows immediately from (SM3.4).  $\square$

*Proof of Proposition 5.10.* The proof is straightforward. If  $f(t) = w(t)g(t) + h(t)$ , then we let  $w \in \ell^2(I)$  be the infinite vector of coefficients of  $g$  and  $h$  with respect to the orthonormal basis  $\{\varphi_n\}_{n \in \mathbb{N}}$ . Note that this vector satisfies

$$\|w\|_2 = \sqrt{\|g\|^2 + \|h\|^2} \leq \|g\| + \|h\|,$$

by Parseval's equality. Also

$$\begin{aligned} f(t) - \mathcal{T}_N w(t) &= f(t) - w(t) \sum_{n=1}^{N/2} \langle g, \varphi_n \rangle \varphi_n(t) - \sum_{n=1}^{N/2} \langle h, \varphi_n \rangle \varphi_n(t) \\ &= g(t) - \mathcal{Q}_N g(t) + w(t) (h(t) - \mathcal{Q}_N h(t)), \end{aligned}$$

and therefore

$$\|f - \mathcal{T}_N w\| \leq \|g - \mathcal{Q}_N g\| + \|w(h - \mathcal{Q}_N h)\| \leq \|g - \mathcal{Q}_N g\| + w_{\max} \|h - \mathcal{Q}_N h\|,$$

as required. □

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